High-frequency asymptotics and 1-D stability of ZND detonations in the small-heat release and high-overdrive limits

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Abstract

We establish one-dimensional spectral, or "normal modes", stability of ZND detonations in the small heat release limit and the related high overdrive limit with heat release and activation energy held fixed, verifying numerical observations of Erpenbeck in the 1960s. The key technical points are a strategic rescaling of parameters converting the infinite overdrive limit to a finite, regular perturbation problem, and a careful high-frequency analysis depending uniformly on model parameters. The latter recovers the important result of high-frequency stability established by Erpenbeck by somewhat different techniques. Notably, the techniques used here yield quantitative estimates well suited for numerical stability investigation.

1 Introduction

In this note, we establish one-dimensional spectral stability in the small heat release and high overdrive limits of ZND detonations, confirming numerical observations of Erpenbeck going back to [Er2], but up to now not rigorously verified. In the process, we recover by a somewhat different argument the fundamental result of Erpenbeck [Er3] that detonations are one-dimensionally stable with respect to sufficiently high frequencies.

The basic argument for stability in the high overdrive limit, based on a strategic rescaling of parameters converting the problem to a small-heat release/small activation energy/maximal shock strength limit on a bounded parameter range, was indicated already in [Z1]. However, the result seems sufficiently fundamental to warrant an exposition giving full detail. In particular, the discussion of [Z1] concerned only bounded frequencies, for which stability follows by a simple continuity argument. Stability for high frequencies can be concluded from a well-known result of [Er3], which, restricted to the one-dimensional setting, asserts that instabilities cannot occur outside a sufficiently large ball. However, the arguments of [Er3], based on semiclassical limit and turning point theory,[1] do not

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[1] Specifically, validation of a formal asymptotic expansion in one over the frequency for complexified x.

readily yield quantitative estimates on rates of convergence or dependence on parameters. As continuous dependence on parameters of the radius outside which instabilities are excluded is crucial for the limiting argument described above, it seems useful to revisit the high-frequency limit problem in greater detail.

Moreover, as pointed out in [CJLW], the issue of matching at $x \to -\infty$ of the formal asymptotic solution with the solution prescribed by the required behavior at spatial infinity appears to require a bit more discussion beyond what is given in [Er3], where it is concluded simply from the observation that the limits as x or frequency go to infinity commute. This argument seems to require either uniform convergence on all of $x \in (-\infty, 0]$ of the formal asymptotic series as frequency goes to infinity, which (to us) does not appear obvious, or else uniform estimates independent of frequency on behavior as $x \to -\infty$.

These aspects (uniform dependence and uniform convergence on $(-\infty, 0]$) of high-frequency behavior are the main issues addressed here, where they are treated by a careful application of the asymptotic ODE techniques developed in [GZ, MeZ1, MaZ3, PZ, Z1]. These in turn are natural outgrowths of the classical asymptotic ODE techniques developed by Coddington, Levinson, Coppel, and others, as described in [CL, Co] and references therein, including those cited by Erpenbeck [Er1, Er2] in describing behavior as $x \to -\infty$.[2] Our arguments are qualitatively different from the ones of Erpenbeck based on semiclassical limit/turning point theory, making use at a key point of the exponential convergence of profiles to a limit as $x \to -\infty$. Specifically, in the neutral case that diagonal elements have uniformly small spectral gap, we apply a variable-coefficient version (Lemma 4.3) of the conjugation lemma of [MeZ1] to close the argument, extending and refining the related constant-coefficient gap lemma estimates of Proposition 5.7, [CJLW], applying to a single mode.[3]

Notably, the simple and concrete estimates thus derived yield quantitative bounds of potential use for numerical stability investigations. Recall [HuZ1] that computational intensity of numerical stability computations increases rapidly with increasing frequency, so that bounds on frequency are of considerable practical interest for applications. See [BZ] for a first effort in this direction in the simplified context of Majda's model [M].

2 Equations and assumptions

The reacting Euler, or Zeldovitch–von Neumman–Doering (ZND) equations commonly used to model combustion, expressed in Lagrangian coordinates, are

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t E + \partial_x (pu) = qk\phi(T)z, \\ \partial_t z = -k\phi(T)z, \end{cases} \tag{2.1}$$

[2]See problem 29, p. 104 of [CL], cited in [Er1].

[3]These estimates, valid on $(-\infty, -C\log|\lambda|]$ for frequency $|\lambda| \to \infty$, do not seem sufficient for our purposes.

where $\tau > 0$ denotes specific volume, u velocity, $E = e + \frac{1}{2}u^2$ specific gas-dynamical energy, $e > 0$ specific internal energy, and $0 \le z \le 1$ mass fraction of the reactant. Here, $k > 0$ measures reaction rate and q heat release of the reaction, with $q > 0$ corresponding to an exothermic reaction and $q < 0$ to an endothermic reaction, while $T = T(\tau, e, z) > 0$ represents temperature and $p = p(\tau, e, z)$ pressure.

The equations are of quasilinear hyperbolic type provided that (but not only when)

$$(p,T) = (p,T)(\tau, e) \quad \text{and} \quad p, p_\tau, T, T_e > 0. \tag{2.2}$$

For simplicity, we assume throughout this paper an ideal gas equation of state and Arrhenius-type ignition function,

$$p = \Gamma \tau^{-1} e, \quad T = c^{-1} e, \quad \phi(T) = e^{-\frac{\mathcal{E}}{T}} \tag{2.3}$$

where $E = e + u^2/2$ is specific (gas-dynamical) energy, $c > 0$ is the specific heat constant, $\Gamma > 0$ is the Gruneisen constant, and $\mathcal{E} \ge 0$ is activation energy. Our results on small heat-release and high-frequency stability clearly extend to the general case (2.2); however, our main results, on the high-overdrive limit, depend in an essential way on the invariances associated with (2.3).

3 Detonation profiles and parametrization

A *right-going strong detonation wave* is a traveling-wave solution

$$(u,z)(x,t) = (\bar u, \bar z)(x - st), \quad \lim_{x\to -\infty} (\bar u, \bar z)(x) = (u_-, 0), \quad (\bar u, \bar z)(x) \equiv (u_+, 1) \text{ for } x \ge 0 \tag{3.1}$$

of (2.1) with speed $s > 0$ connecting a burned state on the left to an unburned state on the right, smooth for $x \le 0$, with a Lax-type gas-dynamical shock at $x = 0$, known as the *Neumann shock.*

Rescaling

$$(x,t,s,\tau,u,T) \to \Big(\frac{\tau_+ s x}{L}, \frac{\tau_+ s^2 t}{L}, 1, \frac{\tau}{\tau_+}, \frac{u}{\tau_+ s}, \frac{T}{\tau_+^2 s^2}\Big), \quad (z,q,k,\mathcal{E}) \to \Big(z, \frac{q}{\tau_+^2 s^2}, \frac{Lk}{\tau_+ s^2}, \frac{\mathcal{E}}{\tau_+^2 s^2}\Big)$$

following [Z1], we may take without loss of generality $s = 1$, $\tau_+ = 1$ and (by translation invariance in u), $u_+ = 0$, leaving $e_+ > 0$ as the parameter determining the Neumann shock.

By explicit computation ([Z1], Appendix C), we have then

$$\bar u = 1 - \bar\tau, \quad \bar e = \frac{\bar\tau(\Gamma e_+ + 1 - \bar\tau)}{\Gamma}, \tag{3.2}$$

$$\bar\tau = \frac{(\Gamma+1)(\Gamma e_+ + 1) - \sqrt{(\Gamma+1)^2(\Gamma e_+ + 1)^2 - \Gamma(\Gamma+2)(1 + 2(\Gamma+1)e_+ - 2q(\bar z - 1))}}{\Gamma + 2}, \tag{3.3}$$

where

$$0 \leq q \leq q_{cj} := \frac{(\Gamma+1)^2(\Gamma e_+ + 1)^2 - \Gamma(\Gamma+2)(1+2(\Gamma+1)e_+)}{2\Gamma(\Gamma+2)}, \tag{3.4}$$

and $\bar{z}' = k\phi(c^{-1}\bar{e}(\bar{z}))\bar{z}$; in the simplest case $\mathcal{E} = 0$, $\bar{z} = e^{kx}$.

The jump at the preceding "Neumann shock" at $x = 0$ is given (see [Z1], App. C) by

$$[\bar{W}] := \Big(1 - \bar{\tau}(0), \bar{\tau}(0) - 1, e_+ - \bar{e}(0), 0\Big)^T. \tag{3.5}$$

Taking finally $k = 1$ by a simultaneous rescaling of x and t if necessary, we can parametrize all possible ZND profiles by

$$(e_+, q, \mathcal{E}, \Gamma), \tag{3.6}$$

where $0 \leq e_+ \leq \frac{1}{\Gamma(\Gamma+1)}$, $0 \leq q \leq q_{cj}(e_+)$, $0 \leq \mathcal{E} < \infty$, and $0 < \Gamma < \infty$. See [Z1] for further details.

3.1 The high-overdrive limit and the scaling of Erpenbeck

A similar scaling was used by Erpenbeck in [Er3], but with e_+ held fixed instead of wave speed s. Converting from Erpenbeck's to our scaling amounts to rescaling the wave speed, so that $T \to T/s^2$ and $\mathcal{E} \to \mathcal{E}/s^2$, and $t \to ts^2$ (u is translation invariant, so irrelevant). Thus, as noted in [Z1], the high-overdrive limit discussed in [Er3], in which $s \to \infty$ with u_+ held fixed, corresponds in our scaling to taking $\mathcal{E} = \mathcal{E}_0 e_+$, $q = q_0 e_+$, and varying e_+ from $e_+ = e_{cj}(q_0)$ (s minimum) to 0 ($s = \infty$), where e_{cj} is determined implicitly by the relation $q_{cj}(e_{cj}) = q_0 e_{cj}$: that is, the simultaneous *zero heat release*, *zero activation energy*, and *strong shock limit* $e_+ \to 0$, $\mathcal{E} \to 0$, and $q \to 0$.

4 Asymptotic ODE theory

4.1 The conjugation lemma

Consider a general first-order system $W' = A^p(x,\lambda)W$, $W \in \mathbb{C}^N$, $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$, with asymptotic limit A^p_- as $x \to -\infty$, where $p \in \mathbb{R}^m$ denote model parameters and $'$ d/dx.

Lemma 4.1 ([MeZ1, PZ])**.** *Suppose for fixed $\theta > 0$ and $C > 0$ that*

$$|A^p - A^p_-|(x,\lambda) \leq Ce^{-\theta|x|} \tag{4.1}$$

for $x \leq 0$ uniformly for (λ, p) in a neighborhood of (λ_0, p_0) and that A varies analytically in λ and continuously in p as a function into $L^\infty(x)$. Then, there exists in a neighborhood of (λ_0, p_0) an invertible linear transformation $P^p(x,\lambda) = I + \Psi^p(x,\lambda)$ defined on $x \leq 0$, analytic in λ and continuous in p as a function into $L^\infty[0, \pm\infty)$, such that

$$|\Psi^p| \leq C_1 e^{-\bar{\theta}|x|} \quad \text{for } x \leq 0, \tag{4.2}$$

for any $0 < \bar\theta < \theta$*, some* $C_1 = C_1(\bar\theta, \theta) > 0$*, and the change of coordinates* $W =: P^p Z$ *reduces* $W' = A^p W$ *to the constant-coefficient limiting system* $Z' = A^p_- Z$ *for* $x \le 0$*.*

Proof. See the proof of Lemma 2.5, [Z4], or Lemma A.1, [Z1]. ☐

4.2 The tracking lemma

Consider an approximately block-diagonal system

$$W' = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}(x,p)W + \delta(x,p)\Theta(x,p)W, \tag{4.3}$$

where Θ is a uniformly bounded matrix, $\delta(x)$ scalar, and p a vector of parameters, satisfying a pointwise spectral gap condition

$$\min \sigma(\Re M_1^p) - \max \sigma(\Re M_2^p) \ge \eta(x) > 0 \text{ for all } x. \tag{4.4}$$

(Here as usual $\Re N := \frac{1}{2}(N + N^*)$ denotes the "real", or symmetric part of N.)

Lemma 4.2 ([MaZ3, PZ, Z1])**.** *Consider a system* (4.3) *under the gap assumption* (4.4)*, with* Θ^p *uniformly bounded and* $\eta \in L^1_{\text{loc}}$*. If* $\sup(\delta/\eta)(x)$ *is sufficiently small, then there exists a unique linear transformation* $\Phi(x,p)$*, possessing the same regularity with respect to* p *as do coefficients* M_j *and* $\delta\Theta$ *(as functions into* $L^\infty(x)$*), for which the graph* $\{(Z_1, \Phi Z_1)\}$ *is invariant under* (4.3)*, and*

$$|\Phi^p(x)| \le C \int_{-\infty}^x e^{\int_y^x -\eta(z)dz}\delta(y)dy \le \sup_{(-\infty,x]} (\delta/\eta). \tag{4.5}$$

Proof. See the proof of Lemma A.4 together with Remark A.6 in [Z1]. ☐

4.3 A variable-coefficient conjugation lemma

The key new technical contribution of this paper at the level of asymptotic ODE is the following simple observation. Consider a first-order system

$$W' = A^p(x;\varepsilon)W := M^p(x;\varepsilon)W + \Theta^p(x;\varepsilon)W, \quad x \le 0, \tag{4.6}$$

$W \in \mathbb{C}^N$, $x \in \mathbb{R}$, $p \in \mathbb{R}^m$, with distinguished parameter $\varepsilon \to 0$, satisfying

$$|\Theta^p(x,\varepsilon)| \le C\varepsilon^2 e^{-\theta\varepsilon|x|}, \tag{4.7}$$

$$|\Re M^p(x,\varepsilon)| \le \varepsilon\delta^p(\varepsilon) + C\varepsilon e^{-\theta\varepsilon|x|} \tag{4.8}$$

for some uniform $C, \theta > 0$, all $x \le 0$, where $\Re M := \frac{1}{2}(M + M^*)$.

Lemma 4.3. *For $\delta^p(\varepsilon) \le \delta_*$ sufficiently small, and $\varepsilon > 0$ sufficiently small, there exists an invertible linear transformation $P^p(x,\varepsilon) = I + \Psi^p(x,\varepsilon)$ defined on $x \le 0$ such that*

$$|\Psi^p| \le C_1 \varepsilon e^{-\theta\varepsilon|x|/2} \quad \text{for } x \le 0, \tag{4.9}$$

and the change of coordinates $W =: P^p Z$ reduces $W' = A^p W$ to $Z' = M^p Z$.

Proof. Equivalently, we construct a solution P^p of the (matrix-valued) homological equation

$$P' = \mathcal{M}^p P + \Theta^p P, \qquad \mathcal{M}P := M^p P - P M^p, \tag{4.10}$$

satisfying $P^p \to I$ as $x \to -\infty$, or, equivalently, a solution Ψ^p of the integral fixed-point equation

$$\mathcal{T}\Psi(x) = \int_{-\infty}^{x} \mathcal{F}^{y\to x}\Theta(y)(I + \Psi(y))dy, \tag{4.11}$$

where $\mathcal{F}^{y\to x}$ is the solution operator of $P' = \mathcal{M}^p P$ from y to x.

Denoting by $(P : Q) := \mathrm{Trace}(\mathrm{P}^*\mathrm{Q})$ the Frobenius inner product, and $\|P\| := (P,P)^{1/2}$ the Frobenius matrix norm, we find that

$$\begin{aligned} \frac{1}{2}\|P\|^2 = \Re(P : P') = \Re(P : M^p P - P M^p) &= (P : (\Re M^p)P - P(\Re M^p)) \\ &\le 2\|\Re M^p\|\|P\|^2, \end{aligned} \tag{4.12}$$

yielding by (4.8) the bound

$$\|\mathcal{F}^{y\to x}\| \le C e^{2\varepsilon\delta^p(\varepsilon)(x-y)} \le C e^{2\varepsilon\delta_*(x-y)}. \tag{4.13}$$

For $\delta_* \le \theta/4$ and $\varepsilon > 0$ sufficiently small, this implies that $\mathcal{T}$ is a contraction on $L^\infty(-\infty,0]$. For, applying (4.7), we have

$$|\mathcal{T}\Psi_1 - \mathcal{T}\Psi_2|_{(x)} \le C\varepsilon^2|\Psi_1 - \Psi_2|_\infty \int_{-\infty}^{x} e^{\theta\varepsilon(x-y)/2} e^{\theta\varepsilon y} dy \le C_1\varepsilon|\Psi_1 - \Psi_2|_\infty e^{-\theta\varepsilon|x|/2}, \tag{4.14}$$

which for ε sufficiently small is less than $\frac{1}{2}|\Psi_1 - \Psi_2|_\infty$.

By iteration, we thus obtain a solution $\Psi \in L^\infty(-\infty,0]$ of $\Psi = \mathcal{T}\Psi$. Further, taking $\Psi_1 = \Psi$, $\Psi_2 = 0$ in (4.14), we obtain, using contraction together with the final inequality in (4.14), that $|\Psi - \mathcal{T}(0)|_{L^\infty(-\infty,x)} \le \frac{1}{2}|\Psi - 0|_{L^\infty(-\infty,x)}$, yielding, as claimed, $|\Psi_{L^\infty(-\infty,x)}| \le 2|\mathcal{T}(0)_{L^\infty(-\infty,x)}| \le 2C_1\varepsilon e^{-\theta\varepsilon|x|/2}$. ☐

5 The Evans–Lopatinski determinant

We now briefly recall the linearized stability theory of [Er1, JLW, Z1, HuZ2]. Shifting to coordinates $\tilde{x} = x - st$ moving with the background Neumann shock, write (2.1) as

$W_t + F(W)_x = R(W)$, where

$$W := \begin{pmatrix} \tau \\ u \\ E \\ z \end{pmatrix}, \quad F := \begin{pmatrix} -u - s\tau \\ \Gamma e/\tau - su \\ u\Gamma e/\tau - sE \\ -sz \end{pmatrix}, \quad R := \begin{pmatrix} 0 \\ 0 \\ qkz\phi(u) \\ -kz\phi(u) \end{pmatrix}. \tag{5.1}$$

To investigate solutions in the vicinity of a discontinuous detonation profile, we postulate existence of a single shock discontinuity at location $X(t)$, and reduce to a fixed-boundary problem by the change of variables $x \to x - X(t)$. In these coordinates, the problem becomes $W_t + (F(W) - X'(t)W)_x = R(W)$, $x \neq 0$, with jump condition $X'(t)[W] - [F(W)] = 0$, $[h(x,t)] := h(0^+,t) - h(0^-,t)$ as usual denoting jump across the discontinuity at $x = 0$.

5.1 Linearization/reduction to homogeneous form

In moving coordinates, $\bar{W}^0$ is a standing detonation, hence $(\bar{W}^0, \bar{X}) = (\bar{W}^0, 0)$ is a steady solution of the nonlinear equations. Linearizing about $(\bar{W}^0, 0)$, we obtain the *linearized equations* $(W_t - X'(t)(\bar{W}^0)'(x)) + (AW)_x = EW$, with jump condition $X'(t)[\bar{W}^0] - [AW] = 0$ at $x = 0$, where $A := (\partial/\partial W)F$, $E := (\partial/\partial W)R$. Computing, we have

(5.2)

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\frac{\Gamma\bar{e}}{\bar{\tau}^2} & -\frac{\Gamma\bar{u}}{\bar{\tau}} & \frac{\Gamma}{\bar{\tau}} & 0 \\ -\frac{\bar{u}\Gamma\bar{e}}{\bar{\tau}^2} & \frac{\Gamma(\bar{e}-\bar{u}^2)}{\bar{\tau}} & \frac{\Gamma\bar{u}}{\bar{\tau}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - sI, \qquad E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{qk\,d\phi(\bar{T})\bar{u}\bar{z}}{c} & \frac{qk\,d\phi(\bar{T})\bar{z}}{c} & qk\phi(\bar{T}) \\ 0 & -\frac{k\,d\phi(\bar{T})\bar{u}\bar{z}}{c} & -\frac{k\,d\phi(\bar{T})\bar{z}}{c} & -k\phi(\bar{T}) \end{pmatrix}.$$

Reversing the original transformation to linear order, following [JLW], by the change of variables $W \to W - X(t)(\bar{W}^0)'(x)$, and noting that x-differentiation of the steady profile equation $F(\bar{W}^0)_x = R(\bar{W}^0)$ gives $(A(\bar{W}^0)(\bar{W}^0)'(x))_x = E(\bar{W}^0)(\bar{W}^0)'(x)$, we obtain modified, *homogeneous* interior equations $W_t + (AW)_x = EW$ together with a modified jump condition accounting for front dynamics of $X'(t)[\bar{W}^0] - [A\big(W + X(t)(\bar{W}^0)'\big)] = 0$.

Lemma 5.1. *For $0 \leq q < q_{cj}$, $A(x) = dF(\bar{W}(x))$ is invertible for all x, with*

$$|A(x) - A_-| \leq Ce^{-\eta|x|} \tag{5.3}$$

for all $x \leq 0$, some $C, \eta > 0$, where $A_- := dF(W_-)$.

Proof. Direct calculation. (The property $\det A_- = 0$ defines q_{cj}, marking the boundary of existence of detonation profiles; see [Z1].) □

5.2 The stability determinant

Seeking normal mode solutions $W(x,t) = e^{\lambda t}W(x)$, $X(t) = e^{\lambda t}X$, W bounded, of the linearized homogeneous equations, we are led to the generalized eigenvalue equations $(AW)' =$

$(-\lambda I + E)W$ for $x \neq 0$, and $X(\lambda[\bar{W}^0] - [A(\bar{W}^0)']) - [AW] = 0$, where "$'$" denotes d/dx, or, setting $Z := AW$, to

$$Z' = GZ, \quad x \neq 0, \tag{5.4}$$

$$X(\lambda[\bar{W}^0] - [A(\bar{W}^0)']) - [Z] = 0, \tag{5.5}$$

with

$$G := (-\lambda I + E)A^{-1}, \tag{5.6}$$

where we are implicitly using the fact that A is invertible, i.e., avoiding the limiting, Chapman–Jouget case $q = q_{CJ}$.

Lemma 5.2 ([Er1, JLW])**.** *For $q \neq q_{CJ}$, on $\mathbb{R}\lambda > 0$, the limiting $(n+1)\times(n+1)$ coefficient matrices $G_\pm := \lim_{z\to\pm\infty} G(z)$ have unstable subspaces of fixed rank: full rank $n+1$ for G_+ and rank n for G_-. Moreover, these subspaces extend analytically to $\mathbb{R}\lambda \leq -\eta < 0$.*

Proof. Straightforward calculation using upper-triangular form of $G_\pm$ [Er1, Er2, Z1, JLW]. □

Corollary 5.3 ([Z1, JLW])**.** *For $q \neq q_{cj}$, On $\mathbb{R}\lambda > 0$, the only bounded solution of* (5.4) *for $x > 0$ is the trivial solution $W \equiv 0$. For $x < 0$, the bounded solutions consist of an (n)-dimensional manifold $\mathrm{Span}\{Z_1^+, \dots, Z_n^+\}(\lambda, x)$ of exponentially decaying solutions, analytic in λ and continuous in parameters $(e_+, q, \mathcal{E}, \Gamma)$, and tangent as $x \to -\infty$ to the subspace of exponentially decaying solutions of the limiting, constant-coefficient equations $Z' = G_- Z$; moreover, this manifold extends analytically to $\mathbb{R}\lambda \leq -\eta < 0$.*

Proof. The first observation is immediate, using the fact that G is constant for $x > 0$, with eigenvalues of positive real part. The second follows from standard asymptotic ODE theory, Lemma 4.1, Appendix 4, together with the fact that G, by Lemma 5.3, decays exponentially to its limit $G_- := G(-\infty)$ as $x \to -\infty$, and that G_- by direct calculation has a single eigenvalue of negative real part for $\Re\lambda > 0$, which extends analytically to $\Re\lambda = 0$ (by spectral separation from the remaining spectra of G_-) and continuously in $(e_+, q, \mathcal{E}, \Gamma)$. □

Definition 5.4. *We define the Evans–Lopatinski determinant*

$$\begin{aligned} D_{ZND}(\lambda) &:= \det\left(Z_1^-(\lambda,0), \quad \cdots, \quad Z_n^-(\lambda,0), \quad \lambda[\bar{W}^0] - [A(\bar{W}^0)']\right) \\ &= \det\left(Z_1^-(\lambda,0), \quad \cdots, \quad Z_n^-(\lambda,0), \quad \lambda[\bar{W}^0] + R(\bar{W}^0)(0^-)\right), \end{aligned} \tag{5.7}$$

where $Z_j^-(\lambda, x)$ are as in Corollary 5.3.

The analytic function D_{ZND} is exactly the *stability function* derived in a different form by Erpenbeck [Er1, Er2]. Evidently (by (5.5) combined with Corollary 5.3), λ is a generalized eigenvalue/normal mode for $\mathbb{R}\lambda \geq 0$ if and only if $D_{ZND}(\lambda) = 0$.

By duality, the zeros of D_{ZND} agree with those of the ajoint formulation

$$\tilde{D}_{ZND}(\lambda) = \langle \tilde{Z}, \lambda[\bar{W}^0] + R\rangle|_{x=0}, \tag{5.8}$$

where $\langle\cdot,\cdot\rangle$ denotes complex inner product and $\tilde{Z}$ denotes an analytically chosen solution of $\tilde{Z}' = -G^*\tilde{Z}$, $x \leq 0$ that is decaying as $x \to -\infty$ (see [HuZ1, CJLW, Z1]).

Definition 5.5. *For $q \neq q_{cj}$ a ZND detonation is spectrally (or "normal modes") stable if the only zero of D_{ZND} (equivalently of $\tilde{D}_{ZND}$) on $\Re\lambda \geq 0$ is a single zero of mulitplicity one at $\lambda = 0$ (necessarily at least multiplicity one by translational invariance).*

6 Continuous dependence and bounded-frequency stability

Proposition 6.1. *For $q \neq q_{cj}$, D_{ZND} and $\tilde{D}_{ZND}$ are analytic in λ and vary continuously in $(e_+, q, \mathcal{E}, \Gamma)$, uniformly on compact subsets of $\{\Re\lambda \geq 0\}$.*

Proof. Immediate, by the construction of the previous section. □

Corollary 6.2. *Under the parametrization* (3.6)*, ZND detonations are stable with respect to bounded frequencies $\{\Re\lambda \geq 0\} \cap \{|\lambda| \leq R\}$,*[4] *any $R > 0$, in the small heat release limit $q \leq q_*$ sufficiently small for Γ, $\mathcal{E}$, e_+ bounded and some $q_* > 0$.*

Proof. By continuity with respect to q, and the properties of uniform limits of analytic functions, the zeros of $\tilde{D}_{ZND}$ on $\{\Re\lambda \geq 0\} \cap \{|\lambda| \leq R\}$ converge as $q \to 0$ to the zeros of $\tilde{D}_{ZND}$ with $q = 0$. Noting that the u and z equations decouple for $q = 0$, and the profile $\bar{u}$ reduces to a gas-dynamical shock, we find readily from (5.8) that $\tilde{D}_{ZND}$ reduces to the Lopatinski determinant for a gas-dynamical shock, which is known (see [Er4, M]) to vanish only at a zero of multiplicity one at $\lambda = 0$. □

Corollary 6.3. *Under the parametrization* (3.6)*, ZND detonations are stable with respect to bounded frequencies $\{\Re\lambda \geq 0\} \cap \{|\lambda| \leq R\}$, any $R > 0$, in Erpenbeck's high-overdrive limit, or, in our scaling, $q, \mathcal{E}, e_+ \to 0$.*

Proof. Immediate, by Corollary 6.2 and the description of Section 3.1. □

7 High-frequency asymptotics and large-frequency stability

7.1 Approximate diagonalization

For q bounded away from q_{cj}, $k = 1$, Γ bounded, suppressing parameters $p = (\Gamma, \mathcal{E}, q, e_+, \hat{\lambda}, \varepsilon)$, rescale $x \to x|\lambda|$, converting the adjoint eigenvalue ODE $\tilde{Z}' = -G^*Z$ to

$$\dot{Z} = B(\varepsilon x)Z + \varepsilon C(\varepsilon x)Z,$$

[4] That is, D_{ZND} (resp. $\tilde{D}_{ZND}$) vanishes on this set only at a zero of multiplicity one at $\lambda = 0$.

where $\varepsilon := |\lambda|^{-1}$, $\hat\lambda := \frac{\lambda}{|\lambda|}$, and $B = \bar{\hat\lambda} A^{-1,T}$, $C = -(EA^{-1})^T$, with A and E as in (5.2).

Noting that $A = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} - I$, where α is the flux Jacobian for ideal gas dynamics, we find from standard gas-dynamical computations that A has distinct eigenvalues $-1, -1, -1 \pm c$, where $c := \frac{\sqrt{\Gamma(\Gamma+1)}}{\tau}$ denotes sound speed, and, for $0 \le q < q_j$, these eigenvalues are uniformly bounded away from zero as well as from each other. By standard matrix perturbation theory [K], there thus exists a smooth coordinate transformation $T = T(B)$ such that $T^{-1}BT$ is diagonal. Making the change of coordinates $Z = TY$, we thus obtain

$$\dot Y = T^{-1}(B + \varepsilon C)TY - T^{-1}\dot T Y =: B_1 Y + \varepsilon C_1 Y, \tag{7.1}$$

where

$$B_1 := T^{-1}BT = \bar{\hat\lambda} \begin{pmatrix} -1+c & 0 & 0 & 0 \\ 0 & -1-c & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{7.2}$$

and, since $\dot T = (\partial T/\partial B)\varepsilon(\partial B/\partial x) = O(\varepsilon e^{-\theta|x|})$ and $|E - E(-\infty)| \le e^{-\theta|x|}$ for $x \le 0$, $\theta > 0$,

$$C_1 := T^{-1}CT - \varepsilon^{-1}T^{-1}\dot T = (T^{-1}CT)(-\infty) + O(e^{-\theta\varepsilon|\hat x|}). \tag{7.3}$$

By block structure of A, we may take without loss of generality $T = \begin{pmatrix} * & * \\ * & I_2 \end{pmatrix}$, whence $(T^{-1}CT)(-\infty) = \begin{pmatrix} 0 & 0 \\ 0 & E_{22}^T \end{pmatrix}$, where $E_{22} = \begin{pmatrix} qk \\ -k \end{pmatrix} (d\phi(\bar T)\bar z, \phi(\bar T))$ is rank one but (for large $\mathcal{E}$, in particular) not always diagonalizable. Nonetheless, at $\hat x = -\infty$, we have

$$E_{22}(-\infty) = \begin{pmatrix} qk \\ -k \end{pmatrix} (0, \phi(T_-)) = \begin{pmatrix} 0 & qk\phi(T_-) \\ 0 & -k\phi(T_-) \end{pmatrix}$$

diagonalizable, so, by a further modification of T_{22}, we may take without loss of generality

$$(T^{-1}CT)_{22}(-\infty) = \begin{pmatrix} 0 & 0 \\ 0 & -k\phi(T_-) \end{pmatrix} \tag{7.4}$$

and $\varepsilon(T^{-1}CT)_{22} = \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & -k\phi(T_-) \end{pmatrix} + O(\varepsilon e^{-\theta|x|})$. Converting briefly back to x-coordinates and applying Lemma 4.1, we thus find that there is a nonsingular coordinate transformation $W = SX$, $|S|, |S^{-1}| \le C$, converting $\dot W = \varepsilon(T^{-1}CT)_{22}W$ to the constant-coefficient equation $\dot X = \varepsilon \begin{pmatrix} 0 & 0 \\ 0 & -k\phi(T_-) \end{pmatrix} X$. Incorporating this further coordinate change in the 2-2 block (only), we obtain, finally

$$\dot X = B_2 X + \varepsilon C_2 X, \tag{7.5}$$

$$
B_2 = \bar{\hat{\lambda}} \begin{pmatrix} -1+c & 0 & 0 & 0 \\ 0 & -1-c & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1-\varepsilon k\phi(T_-)/\bar{\hat{\lambda}} \end{pmatrix}, \quad C_2 = \begin{pmatrix} O(e^{-\theta\varepsilon|\hat{x}|} & O(e^{-\theta\varepsilon|\hat{x}|} \\ O(e^{-\theta\varepsilon|\hat{x}|} & 0_2 \end{pmatrix}, \tag{7.6}
$$

where O_2 denotes the 2×2 zero matrix.

Remark 7.1. *The reduction just performed, using the conjugation lemma to diagonalize the lower righthand block, is a delicate point of the analysis, avoiding potential difficulties associated with turning points where E_{22} becomes nondiagonalizable.*[5]

Using again standard matrix perturbation theory, it follows for ε sufficiently small that there is a further smooth coordinate transformation

$$
Q = \begin{pmatrix} 1 & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) \\ O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & 1 & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) \\ O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & 1 & 0 \\ O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & 0 & 1 \end{pmatrix}
$$

such that $M := Q^{-1}(B_2 + \varepsilon C_2)Q$ is block-diagonal,

$$
M = \begin{pmatrix} \bar{\hat{\lambda}}(-1+c) + O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & 0 & 0 & 0 \\ 0 & \bar{\hat{\lambda}}(-1-c) + O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & 0 & 0 \\ 0 & 0 & -\bar{\hat{\lambda}} & 0 \\ 0 & 0 & 0 & -\bar{\hat{\lambda}} - \varepsilon k\phi(T_-) \end{pmatrix}, \tag{7.7}
$$

and $\Theta := -Q^{-1}\dot{Q} = O(\varepsilon^2 e^{-\theta\varepsilon|\hat{x}|})$, taking the equations to the approximate block-diagonal form treated in Lemmas 4.2 and 4.3, of

$$
W' = MW + \Theta W. \tag{7.8}
$$

7.2 High-frequency stability

Proposition 7.2. *For $k = 1$, Γ, $\mathcal{E}$ bounded, and q bounded away from q_{cj}, ZND detonations are stable with respect to sufficiently high frequencies; that is, there exists $R > 0$ independent of $(\Gamma, \mathcal{E}, q, e_+)$ such that $D_{ZND}(\lambda) \neq 0$ for $\Re\lambda \geq 0$ and $|\lambda| \geq R$.*

Proof. Case (i) ($\Re\lambda << \varepsilon|\lambda|$) Equivalently, $\Re\hat{\lambda} << \varepsilon$, whence, in (7.7), there is a spectral gap between the fourth diagonal entry, $-\bar{\hat{\lambda}} - \varepsilon k\phi(T_-)$, which has real part $\leq -\varepsilon\eta$ for $\eta = \phi(T-) > 0$, and the first three diagonal entries, $\bar{\hat{\lambda}}(-1 \pm c)$ and $-\bar{\hat{\lambda}}$, which have real parts $\geq -C\Re\bar{\hat{\lambda}} >> -\varepsilon$.

[5] Unimportant for $|\lambda| >> 1 + \mathcal{E}$, these dominate behavior in the high-activation energy limit $\mathcal{E} \to \infty$ [BZ, Z5].

Applying Lemma 4.2 to (7.8), we find that there is a graph $W_4 = \Phi(W_1, W_2, W_3)$, $\Phi = O(\varepsilon)$ that is invariant under (7.8), from which we may reduce to an equation

$$\check{W}' = \check{M}\check{W} + \check{\Theta}\check{W} \tag{7.9}$$

on $\check{W} = (W_1, W_3, W_3)$ alone, with

$$\check{M} = \begin{pmatrix} \bar{\hat{\lambda}}(-1+c) + O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & 0 & 0 \\ 0 & \bar{\hat{\lambda}}(-1-c) + O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & 0 \\ 0 & 0 & -\bar{\hat{\lambda}} \end{pmatrix} \tag{7.10}$$

satisfying (4.8) with $\delta_* << \varepsilon$ and $\check{\Theta} = O(\varepsilon^2 e^{-\theta\varepsilon|\hat{x}|})$.

Applying now Lemma 4.3 to (7.9), we find that there is a coordinate transformation $\check{W} = PX$, $P = I + O(\varepsilon)$, taking (7.9) to the block-decoupled equation $\dot{X} = \check{M}X$, of which the unique (up to constant multiplier) solution X^- decaying as $\hat{x} \to -\infty$ has value $X^-(0)$ at $\hat{x} = 0$ parallel to $(1, 0, 0,)^T$. Untangling coordinate changes, we find that unique (up to constant multiplier) solution W^- of (7.8) decaying as $\hat{x} \to -\infty$ is parallel to $I + O(\varepsilon)$ times $(1, 0, 0, 0)^T$ and the unique (up to constant multiplier) solution $\tilde{Z}^-(0)$ of adjoint eigenvalue equation $\tilde{Z}' = -G^* Z$ is parallel to $I + O(\varepsilon)$ times the left unstable eigenvector of $A^{-1}(0)$, or $(\ell, 0)^T$, where ℓ is the left unstable eigenvector of the gas-dynamical flux Jacobian $\alpha(0)$.

Likewise, $\lambda[\bar{W}] + R(\bar{W}(0^-))$ is parallel to $I + O(\varepsilon)$ times $[\bar{W}] = ([u], 0)^T$, whence, combining these facts, we find that $\hat{D}_{ZND}(\lambda) = \tilde{Z}^-(0) \cdot (\lambda[\bar{W}] + R(\bar{W}(0^-)))$ is proportional to $1 + O(\varepsilon)$ times $\Delta(\lambda) := \ell \cdot \lambda[u]$, which may be recognized as the Lopatinski determinant for an ideal gas-dynamical shock, known by [Er4, M] to be nonvanishing on $\Re\lambda \geq 0$ except at $\lambda = 0$, with all constants uniform in model parameters and $\Re\lambda \geq 0$. For $|\lambda|$ sufficiently large, therefore, or equivalently, $\varepsilon := |\lambda|^{-1}$ sufficiently small, we find that $\hat{D}_{ZND}(\lambda) \neq 0$.

Case (ii) ($\Re\lambda \geq C^{-1}\varepsilon|\lambda|$) Equivalently, $\Re\hat{\lambda} \geq C^{-1}\varepsilon$, whence, applying Lemma 4.2 to (7.8), we find that there is a graph $(W_2, W_3, W_4) = \Phi(W_1)$, $\Phi = O(\varepsilon)$ that is invariant under (7.8), from which we find that the unique unique (up to constant multiplier) solution W^- of (7.8) decaying as $\hat{x} \to -\infty$ has value $W^-(0)$ at $\hat{x} = 0$ parallel to $I + O(\varepsilon)$ times $(1, 0, 0, 0)^*$. Untangling coordinate changes, and arguing as in the previous case, we thus find again that $\hat{D}_{ZND}(\lambda) = \tilde{Z}^-(0) \cdot (\lambda[\bar{W}] + R(\bar{W}(0^-)))$ is proportional to $1 + O(\varepsilon)$ times the gas-dynamical Lopatinski determinant $\Delta(\lambda) := \ell \cdot \lambda[u]$, $\ell \cdot [u] \neq 0$, hence nonvanishing for $|\lambda|$ sufficiently large. $\square$

Remark 7.3. Applying Lemmas 4.2 and 4.3 in sequence in this way, one may treat the situation arising in the multi-dimensional case (see [Er3]) of an approximately block-diagonal system for which some blocks have a uniform spectral gap and others have uniformly small spectral gap. We hope to report on this in future work.

Remark 7.4. Though we did state it, the arguments above show that there exists a change of coordinates $Q = I + O(\varepsilon e^{-\theta\varepsilon|\hat{x}|})$ taking (7.8) to exactly diagonal form $W' = MW$,

where M is as in (7.7).[6] This gives information about the full flow, and not only the decaying solution important for the stability theory. Note that, in the exactly diagonal coordinates W, the first two entries correspond to coefficients of the first two eigenvectors of A in the eigenexpansion of $\tilde{Z}$, while the second two entries correspond to unknown linear combinations of the the coefficients of the third and fourth eigenvectors of A.[7]

Likewise, a closer look at the proof reveals the asymptotic description

$$D_{ZND}(\lambda) = e^{C_1\lambda + C_0}\Delta(\lambda)(1 + O(\varepsilon)), \tag{7.11}$$

where $\Delta(\lambda) := \ell\cdot\lambda[u]$, $\ell\cdot[u] \neq 0$ is the Lopatinski determinant associated with the Neumann shock. This can be used as in [HLyZ1] as the basis of a convergence study, to obtain practical bounds on unstable eigenvalues. Higher order approximants

$$D_{ZND}(\lambda) = e^{C_1\lambda + C_0 + C_{-1}\lambda^{-1}}\Delta(\lambda)(1 + D_1\varepsilon + O(\varepsilon^2)),$$

etc., may be obtained by further diagonalizations as detailed in [MaZ3].

Corollary 7.5. *Under the parametrization* (3.6)*, ZND detonations are stable in both the small-heat release and (with Erpenbeck's scaling, Section 3.1) high-overdrive limits.*

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A Numerical implementation

For purpose of numerical applications, we provide also a simpler high-frequency argument requiring the weaker estimate

$$\Phi = \begin{pmatrix} O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon^2 e^{-\theta\varepsilon^2|\hat{x}|}) & O(\varepsilon^2 e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon^2 e^{-\theta\varepsilon^2|\hat{x}|}) \\ O(\varepsilon^2 e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) \\ O(\varepsilon^2 e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) \\ O(\varepsilon^2 e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) & O(\varepsilon e^{-\theta\varepsilon|\hat{x}|}) \end{pmatrix} \tag{A.1}$$

rather than $\Theta = O(\varepsilon^2 e^{-\theta\varepsilon|\hat{x}|})$ in (7.8), removing the need for the intermediate coordinate transformation S in the lower righthand 2×2 block. This avoids an abstract conjugation step that is difficult to estimate efficiently numerically, and also provides a slightly simpler proof treating all frequencies at once instead of dividing into cases. On the other hand, it provides information only about the single mode $\tilde{Z}^-$ associated with the stability determinant, and not the entire flow of the adjoint eigenvalue ODE, which may be of interest in more general situations.

[6] This follows by separating off scalar diagonal entries with spectral gap from other entries using Lemma 4.2, to obtain scalar equations $w' = mw + \theta w$ with $|\theta| = O(\varepsilon^2 e^{-\theta\varepsilon|\hat{x}|})$ for which error θ can be shown to be negligible by explicit exponentiation. (Conjugation of the nonscalar block (7.9) has already been shown.)

[7] Recall, these depend on the abstract conjugation prescribed in going from (7.2) to (7.6).

A.1 Variable-coefficient gap lemma

Consider a first-order system (4.6) satisfying (4.7) and

$$\Re M^p(x,\varepsilon) \le \varepsilon\delta^p(\varepsilon) + C\varepsilon e^{-\theta\varepsilon|x|} \tag{A.2}$$

for some uniform $C, \theta > 0$, all $x \le 0$, $\Re M := \frac{1}{2}(M + M^*)$. (Note, in contrast with (4.8), that this is a bound from above only.) Assume, further, that there exists a smooth vector $V_*^p(x)$, $1/C \le |V_*^p| \le C$, for which $M^p V_*^p \equiv 0$.

Lemma A.1. *Assuming* (A.2)*, for $\delta^p(\varepsilon) \le \delta_*$ sufficiently small, and $\varepsilon > 0$ sufficiently small, there exists a solution $V^p(x,\varepsilon)$ of* (4.6) *defined on $x \le 0$ such that*

$$|(V^p - V_*^p)(x)| \le C_1 \varepsilon e^{-\theta\varepsilon|x|/2} |V_*^p| \quad \text{for } x \le 0. \tag{A.3}$$

Proof. We seek, equivalently, a solution V^p of the integral fixed-point equation

$$\mathcal{T}V(x) = V_*^p(x) + \int_{-\infty}^{x} \mathcal{F}^{y\to x}\Theta(y)V(y)dy, \tag{A.4}$$

where $\mathcal{F}^{y\to x}$ is the solution operator of $V' = M^p V$ from y to x. From (A.2), we obtain by an easy energy estimate like that of (4.12) the bound

$$\|\mathcal{F}^{y\to x}\| \le Ce^{\varepsilon\delta^p(\varepsilon)(x-y)} \le Ce^{\varepsilon\delta_*(x-y)} \ \text{ for } \ x > y. \tag{A.5}$$

For $\delta_* \le \theta/2$ and $\varepsilon > 0$ sufficiently small, this implies that $\mathcal{T}$ is a contraction on $L^\infty(-\infty, 0]$. For, applying (4.7) and (A.5), we have

$$|\mathcal{T}V_1 - \mathcal{T}V_2|_{(x)} \le C\varepsilon^2 |V_1 - V_2|_\infty \int_{-\infty}^{x} e^{\theta\varepsilon(x-y)/2} e^{\theta\varepsilon y} dy \le C_1 \varepsilon |V_1 - V_2|_\infty e^{-\theta\varepsilon|x|/2}, \tag{A.6}$$

which for ε sufficiently small is less than $\frac{1}{2}|V_1 - V_2|_\infty$. By iteration, we thus obtain a solution $V \in L^\infty(-\infty, 0]$ of $V = \mathcal{T}V$. Further, taking $V_1 = V$, $V_2 = 0$ in (4.14), we obtain, using contraction together with the final inequality in (A.6), that $|V - V_*|_{L^\infty(-\infty,x)} = |V - \mathcal{T}(0)|_{L^\infty(-\infty,x)} \le \frac{1}{2}|V - 0|_{L^\infty(-\infty,x)}$, yielding (A.3) as claimed. ☐

A.2 Alternate high-frequency analysis

Alternate proof of Prop. 7.2. By inspection, $\mathcal{M} := (M + \check\Theta) - (M + \check\Theta)_{11} I$ satisfies (A.2) for M as in (7.7), $\varepsilon := |\lambda|^{-1}$, and

$$\check\Theta = \begin{pmatrix} \Theta_{11} & 0 & 0 & 0 \\ 0 & \Theta_{22} & \Theta_{23} & \Theta_{24} \\ 0 & \Theta_{32} & \Theta_{33} & \Theta_{34} \\ 0 & \Theta_{42} & \Theta_{43} & \Theta_{44} \end{pmatrix}$$

with Θ as in (A.1), with $\mathcal{M}V_* \equiv 0$ for $V_* := (1,0,0,0)^T$, whence, applying Lemma A.1, we obtain a decaying solution $W(x;\varepsilon) = \omega(x) e^{\hat{\bar\lambda}(-1+c)\hat x} V(x;\varepsilon)$ of (7.8) converging as $O(\varepsilon)$ in

relative error to $(1,0,0,0)^T$, where $\omega(\hat{x}) = e^{\int_{-\infty}^{\hat{x}} O(\varepsilon e^{-\theta\varepsilon|\hat{y}|}d\hat{y}} > 0$ is uniformly bounded above and below. Untangling coordinate changes, and noting that $\lambda[\bar{W}] + R(\bar{W}(0^-))$ is parallel to $I + O(\varepsilon)$ times $[\bar{W}] = ([u], 0)^T$, we thus find that $\hat{D}_{ZND}(\lambda) = \tilde{Z}^-(0) \cdot (\lambda[\bar{W}] + R(\bar{W}(0^-)))$ is proportional to $1 + O(\varepsilon)$ times the gas-dynamical Lopatinski determinant $\Delta(\lambda) := \ell \cdot \lambda[u]$, $\ell \cdot [u] \neq 0$, hence nonvanishing for $|\lambda|$ sufficiently large. $\square$

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